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Recurrence of the Eigenstates of a Schrödinger Operator with Automatic Potential

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We consider the Schrödinger eigenvalue problem in the discrete case with a potential assuming two values distributed according to the automatic sequence of Prouhet–Thue–Morse. We show that there are no localized states and that the generalized eigenvectors are recurrent on a geometrical set stemming from the hierarchical nature of the potential.

KEY WORDS: Thue–Morse sequence; Schrödinger; spectral analysis; localization; critical states.

1. INTRODUCTION

We study a one-dimensional discrete Schrödinger operator whose potential assumes two values distributed according to the Prouhet–Thue–Morse sequence. Various such one-dimensional models have been investigated, namely with random or almost periodic potential. In addition, much work has been devoted to studying the Schrödinger operator with a potential generated by the Fibonacci sequence, which can be seen, as well as a quasiperiodic sequence, as a sequence generated by a substitution (as is the Prouhet–Thue–Morse one). The discovery of quasicrystals[(1)] gives physical grounds for the study of such a potential which can be thought of as modeling 1D quasicrystals. With respect to the Fourier transform, the Fibonacci and the Prouhet–Thue–Morse sequences have very different properties; indeed, the latter has a singular continuous measure as Fourier transform. One problem is to determine whether such operators have localized or extended eigenstates. In the Fibonacci case, Kohmoto *et al.*, in

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ref. 2 and subsequent articles, pointed out certain nonlocalized states which they called critical and which have been rigorously worked out by Süto.[3] The present work completes previous articles.[4,5] In ref. 5 it is rigorously proved that, in this case, there exist states extended in a very strong way: they assume any of their values on an automatic set of integers of positive density. But these states are associated with very special points of the spectrum. In this work, we show that the operator occurring in the Prouhet–Thue–Morse case has no eigenvalue in $l^2(Z)$, which means that there are no localized states (the opposite of the disordered case). Furthermore, our proof shows that any eigenstate is not too small on a geometric progression.

2. NOTATIONS AND RESULTS

Let us consider the following iterative construction:

$$0\,|\,1$$

$$0110\,|\,1001$$

$$0110100110010110\,|\,1001011001101001$$

where each line is obtained from the preceding one by replacing 1's by 1001 and 0's by 0110. Lines are numbered starting from 0. Elements of the nth line are numbered from -4^n to 4^n-1, so that the terms of rank 0 immediately follow the vertical line. It is to be noticed that each line can be obtained by adding prefixes and suffixes to the preceding one. Therefore, this construction converges toward a sequence $\{\varepsilon_j\}_{j\in Z}$ of zeros and ones.

On the other hand, the nth line is the first half of the word $\sigma^{2n+1}(0)$, shifted leftward by 4^n positions, where σ is the Thue–Morse substitution. In other words, the sequence ε is one of those which are considered in ref. 5.

It results from its construction that, for any $n \geqslant 0$, the sequence ε has the following structure:

$$\ldots\sigma^{2n+1}(0)\,|\,\sigma^{2n+1}(1)\,\sigma^{2n+1}(0)\,\sigma^{2n+1}(0)\,\sigma^{2n+1}(1)\,\sigma^{2n+1}(0)\ldots \tag{1}$$

Now, let q_0 and q_1 be two distinct real numbers and H the following Schrödinger operator:

$$(Hx)_n = x_{n+1} + x_{n-1} - q_{\varepsilon_n} x_n \tag{2}$$

acting on the Hilbert space $l^2(Z)$.

The spectrum of this operator has been studied in ref. 5 and numerical evidence has been given that it is of zero Lebesgue measure. In addition, it has been rigorously proved that, for a dense subset of points in the spectrum, the corresponding states are extended. As a matter of fact, these states not only do not vanish at infinity, but merely assume their maximum value on a relatively dense set of integers. In this work, we continue the study of the localization of modes. We prove that they all are extended. More precisely, we prove the following result:

Theorem. The operator H has no eigenvectors in the space $c_0(Z)$ of sequences vanishing at infinity.

This result and its proof are to be put together with ref. 6.

Proof. The proof will proceed by *reductio ad absurdum*. So, we suppose that we have a real number λ and a nonzero sequence $\{x_n\}_{n\in Z}$ such that $\lim_{n\to\infty} x_n = 0$ and

$$\lambda x_n = x_{n+1} + x_{n-1} - q_{\varepsilon_n} x_n \qquad \text{for} \quad n \in Z \tag{3}$$

This recursion relation can also be written in the following way:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda - q_{\varepsilon_n} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}$$

or, equivalently,

$$u_{n+1} = M_{\varepsilon_n} u_n = M^{(n)} u_0 \tag{4}$$

with obvious notations.

Then by (1) we have

$$u_{2^{2n+1}} = M^{(2^{2n+1})} u_0 \qquad \text{and} \qquad u_0 = M^{(2^{2n+1})} u_{2^{2n+1}} \tag{5}$$

The assumption that x is a nonzero sequence exactly means $u_0 \neq 0$.

Let us define two sequences of matrices A_n and B_n by recursion:

$$A_0 = M_1 M_0 \tag{6}$$

$$B_0 = M_0 M_1 \tag{7}$$

$$A_{n+1} = A_n B_n B_n A_n \tag{8}$$

$$B_{n+1} = B_n A_n A_n B_n \qquad \text{for} \quad n \geqslant 0 \tag{9}$$

Then $A_n = M^{(2^{2n+1})}$.

Thus, if u_n goes to zero at infinity, we have

$$\lim_{n\to\infty} u_{2^{2n+1}} = \lim_{n\to\infty} A_n u_0 = 0 \tag{10}$$

$$\lim_{n\to\infty} u_{-2^{2n+1}} = \lim_{n\to\infty} A_n^{-1} u_0 = 0 \tag{11}$$

$$\lim_{n\to\infty} u_{2^{2n+2}} = \lim_{n\to\infty} B_n A_n u_0 = 0 \tag{12}$$

$$\lim_{n\to\infty} u_{5.2^{2n+1}} = \lim_{n\to\infty} B_n A_n B_n^2 A_n u_0 = 0 \tag{13}$$

We will prove that Eqs. (10)–(13) cannot be fulfilled. Let us first recall some properties of the sequences A_n and B_n.

Proposition 1. Let us set $t_n = \operatorname{Tr} A_n$; then

$$t_n = \operatorname{Tr} B_n \tag{14}$$

$$t_{n+1} = t_n^2 \operatorname{Tr}(A_n B_n) - 2t_n^2 + 2 \tag{15}$$

$$A_n = B_n + \mu_n T, \qquad \text{where} \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{16}$$

Proof. Equation (14) follows immediately from the definitions (6)–(9). Equation (15) is well known and can be obtained by taking the trace of Eq. (8) and using that A_n and B_n satisfy their characteristic equation. Equation (16) is true for $n = 0$ with $\mu_0 = q_1 - q_0$; then by recursion we have to prove that if two unimodular $2 * 2$ matrices A and B satisfy

$$A = B + \mu T$$

$$\operatorname{Tr} A = \operatorname{Tr} B = t$$

then

$$ABBA - BAAB = \mu' T \tag{17}$$

Indeed,

$$\begin{aligned} ABBA - BAAB &= t(ABA - BAB) - A^2 + B^2 \\ &= t(ABA - BAB) - tA + tB \end{aligned} \tag{18}$$

and

$$\begin{aligned} ABA - BAB &= A(A - \mu T)A - (A - \mu T)\,A(A - \mu T) \\ &= -\mu ATA + \mu TA^2 + \mu A^2 T - \mu^2 TAT \\ &= -\mu ATA + \mu t TA + \mu t AT - \mu^2 TAT - 2\mu T \end{aligned} \tag{19}$$

Since A and B are unimodular, we have

$$A^2 - tA + 1 = 0$$

$$B^2 - tB + 1 = 0$$

that is,

$$(A - \mu T)^2 - t(A - \mu T) + 1 = 0$$

Thus

$$\mu(AT + TA) - t\mu T - \mu^2 = 0 \tag{20}$$

and (20), (19), and (18) easily provide the result.

Lemma 2. μ_n, as defined in Proposition 1, satisfies

$$\mu_n^2 = \mu_n \operatorname{Tr}(TA_n) \qquad \text{and} \qquad \operatorname{Tr}(B_n A_n) = t_n^2 - \mu_n^2 - 2 \tag{21}$$

Proof. The first relation is obtained by taking the trace of (20) and the second one by direct calculation of $\operatorname{Tr}(B_n A_n) = \operatorname{Tr}[(A_n - \mu_n T) A_n]$.

We now can resume the proof of the theorem. First, by (10) and (11) we have

$$\lim_{n \to \infty} (u_{2^{2n+1}} + u_{-2^{2n+1}}) = t_n u_0 = 0$$

Thus,

$$\lim_{n \to \infty} t_n = 0 \tag{22}$$

This relation together with (15) and Lemma 2 yields

$$\lim_{n \to \infty} t_n^2 \mu_n^2 = 2 \tag{23}$$

Remark. In particular, this implies that $\lim_{n \to \infty} \mu_n^2 = +\infty$. Thus, $\mu_n \neq 0$ for all n and the first part of Lemma 2 can be set as $\mu_n = \operatorname{Tr}(TA_n)$.

Then (12) provides

$$\lim_{n \to \infty} (A_n - \mu_n T) A_n u_0 = 0$$

i.e.,

$$\lim_{n \to \infty} (t_n A_n u_0 - u_0 - \mu_n TA_n u_0) = 0$$

Hence

$$\lim_{n\to\infty} \mu_n T A_n u_0 = -u_0 \tag{24}$$

Now, let us turn our attention to (13):

$$\begin{aligned} B_n A_n B_n^2 A_n &= B_n A_n (t_n B_n - 1) A_n \\ &= t_n \operatorname{Tr}(B_n A_n) B_n A_n - t_n - t_n B_n A_n + B_n \end{aligned}$$

Thus (13) yields

$$\begin{aligned} 0 &= \lim_{n\to\infty} [t_n \operatorname{Tr}(B_n A_n) B_n A_n u_0 + B_n u_0] \\ &= \lim_{n\to\infty} [t_n \operatorname{Tr}(B_n A_n) B_n A_n u_0 - \mu_n T u_0] \end{aligned}$$

A fortiori, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} [t_n^2 \operatorname{Tr}(B_n A_n) B_n A_n u_0 - \mu_n^2 T u_0] \\ &= \lim_{n\to\infty} [t_n^2(-\mu_n^2 - 2) B_n A_n u_0 - \mu_n^2 T u_0] \end{aligned}$$

which, taking into account Lemma 2 and (23), gives $Tu_0 = 0$, a contradiction.

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